Self-Similarity Parameter Estimation for K-Dimensional Processes

S. Bianchi, A. M. Palazzo, A. Pantanella, and A. Pianese

I. INTRODUCTION

A distinctive feature of fractals, both deterministic and stochastic, is self-similarity, that is the property they display to be at some degree scale invariant under proper renormalization. This notion is often used to describe the behaviour of many phenomena, such as e.g. complex networks [1], internet applications [2]-[5] turbulence [6], geophysical record [7], [8], economics and finance [9]-[12], biology [13], image, object detection and video filtering [14], optics [15].

The number of fields in which self-similarity is claimed to occur has motivated many contributions on the estimation problem (see, e.g., [16], [17] for a survey). Whereas the notion of (strong) self-similarity is given in terms of the equality

\[ \{ X(at_1), X(at_2), \ldots, X(at_k) \} \overset{d}{=} \{ a^{H_0} X(t_1), a^{H_0} X(t_2), \ldots, a^{H_0} X(t_k) \} \]

holds for its finite-dimensional distributions. A recent reference work in this field is [20].

Definition 1. The real-valued, continuous time stochastic process \( X(t), t \in T \) is self-similar with index \( H_0 > 0 \) (shortly, \( H_0 \)-ss) if, for any \( a \in \mathbb{R}^+ \) and integer \( k \) such that \( t_1, \ldots, t_k \in T \), the equality

\[ \{ X(at_1), X(at_2), \ldots, X(at_k) \} \overset{d}{=} \{ a^{H_0} X(t_1), a^{H_0} X(t_2), \ldots, a^{H_0} X(t_k) \} \]

holds for its finite-dimensional distributions.

Definition 2. The second-order stationary, real-valued stochastic process \( X(t) \) is \( H_0 \)-second order self-similar if – denoted by \( Y(t,a) = X(t+a) - X(t) \) it’s a lagged increments, and by \( \overline{Y}(t,m) = m^{-1} \sum_{i=1}^{m} Y(t,i) \) the averaged (over blocks of length \( m \)) sequence – it holds

\[ \text{Var}(\overline{Y}(t,m)) = m^{2(H_0-1)} \text{Var}(Y(t,1)) \]

The process is also said \( H_0 \)-second order asymptotically self-similar if, for \( k \in \{1,2,\ldots\} \)

\[ \text{Var}(\overline{Y}(t,km)) = k^{2(H_0-1)} \text{Var}(\overline{Y}(t,m)) \]

as \( m \) diverges.

Example. A reference process in self-similarity is the fractional Brownian motion (fBm). Originally defined in a seminal paper by [21], the one-dimensional fBm (in notation, \( B_H(t) \)) is the only centered Gaussian, \( H \)-sssi\(^1\) stochastic process with autocovariance function

\[ E[B_H(t)B_H(s)] = \frac{K^2}{2} \left| t \right|^{2H} + \left| s \right|^{2H} - \left| t-s \right|^{2H} \]

where \( K^2 = \text{Var}(B_H(1)) \) and \( t, s \in \mathbb{R}^+ \). From (3) it readily follows that

\[ E|B_H(t) - B_H(s)|^2 = K^2 |t-s|^{2H} \]

\(^1\) H-sssi stands for H-self similar with stationary increments.
The stochastic process $B = \{B_{H_i}(t)^k, \ t \in [0, +\infty)\}$ where $B_{H_i}$ ($i = 1, \ldots, k$) are $k$ independent copies of the (one-dimensional) fBm with the same self-similarity parameter $H \in [0,1]$, is named $k$-dimensional fBm. Although more general definitions for the $k$-dimensional fBm can be found in literature, we will restrict our simulations to the case when the self-similarity parameter is the same along all the directions; obviously, the algorithm continues working even in the general case $H_i \neq H_j$ for $i \neq j$. The parameter $H$ affects the smoothness of the process, as shown in the examples provided – for the 2-dimensional case – by Figures 1-3.

The process displays smoother and smoother surfaces as $H$ grows. The value $H = \frac{1}{2}$ recovers the Brownian motion as a special case.

Denoted by $E(\cdot)$ the expected value, it is easy to check that equality (1) implies

$$
E[|X(t)|^q] \sim t^{H/q}E[|X(1)|^q]
$$

which justifies the fact that self-similarity is often tested through the scaling behaviour of the process sample moments.

Nonetheless, several problems arise with this approach: (a) as relation (4) does not imply relation (3), the (4)-based conclusions can be questionable; (b) generally, relation (4) is studied only for particular values of $q$ (1 or 2 are the most frequent cases), what leads to infer weak forms of self-similarity (e.g., second-order or asymptotical self-similarity).

In order to bypass these problems and test the condition of self-similarity in its larger meaning (that is Definition 1), [18] suggests a different method which takes into account the whole process distribution.

The method is shortly summarized in the followings. Let $\mathcal{A}$ be any bounded set of $R^+$ and let $\underline{a} = \min(\mathcal{A})$ and $\overline{A} = \max(\mathcal{A})$. For any $a \in \mathcal{A}$, the set $\{X(at)\}$ of the $a$-lagged rescaled process is considered. Denoted by $\Phi$ the $k$-dimensional distribution of $X$ and setting

$$
\Phi_{aX}(x) = \Phi_{aX(1)}(x), \ x \in R^k
$$

When $X(t)$ is $H_\rho$-self-similar, one has

$$
\Phi_{aX(t)}(x) = \Pr(a^{-H}X(at) < x_1, \ldots, a^{-H}X(at) < x_k) = \\
\Pr(a^{H-H}X(at_1) < x_1, \ldots, a^{H-H}X(at_k) < x_k) = \\
\Phi_{aX(1)}(x) = \\
\Pr(X(t_1) < a^{H-H}x_1, \ldots, X(t_k) < a^{H-H}x_k) = \\
\Phi_{aX(1)}(a^{H-H}x).
$$

Therefore, denoted by $\Psi_H = \{\Phi_{aX(1)}(x), a \in \mathcal{A}\}$ the set of the (absolutely continuous) $k$-dimensional probability distribution functions of $\{a^{-H}X(at)\}$, one can define as distance function $\rho$ the one induced by the sup-norm $\| \cdot \|_\infty$ and assume as measure of the discrepancy among the rescaled distributions the diameter $\delta$ of the metric space $(\Psi_H, \rho)$. Namely:

$$
\delta^\Psi(\Psi_H) = \sup_{x \in R^k} \sup_{a, i, j} \left| \Phi_{a^{-H}X(i)}(x) - \Phi_{a^{-H}X(j)}(x) \right|
$$

For the diameter, in [18] the following three propositions are proved.

Proposition 1. $\{X(t), t \in T\}$ is $H_\rho$-ss if and only if, for any
bounded $A \subset \mathbb{R}^n$ and any integer $k$, $\delta^k\left(\Psi_{H_0}\right) = 0$.

Proposition 2. Let $\{X(t), t \in T\}$ be $H_0$-ss, $\overline{A} > a$ and $X \geq 0 \ (x \leq 0)$. Then $\delta^k\left(\Psi_H\right)$ is non-increasing for $H \leq H_0$ and non-decreasing for $H \geq H_0$.

Proposition 3. Let $\{X(t), t \in T\}$ be $H_0$-ss, $x \geq 0 \ (x \leq 0)$ and let $\{A_p\}_{p=1}^{\infty}$ be a sequence of sets such that $A_p \leq A_q$ for $p < q$. Then, with respect to the sequence $\{A_p\}$, the diameter $\delta^k\left(\Psi_H\right)$ is: (i) non-decreasing if $H \neq H_0$; (ii) an identically zero function if $H = H_0$.

Proposition 1 basically states the uniqueness of the self-similarity parameter in terms of the function $\delta^k\left(\Psi_H\right)$. Proposition 2 provides a necessary condition of self-similarity, requiring the diameter to be a monotone function of $H$ (non increasing for $H \leq H_0$ and non decreasing for $H \geq H_0$). Finally, Proposition 3 states that the diameter is monotone also with respect to an increasing sequence of lags. Exploiting the three propositions, one can test for self-similarity simply by estimating the minimum of the function with respect to $H$, once a minimal and a maximal lag have been fixed. Further propositions are deduced in order to assess the statistical significance of $\delta^k$ by the well-known Kolmogorov-Smirnov test, but here we just want to focus on the estimation of $H_0 = \arg \min_H \delta^k$.

Fig. 4 displays an application of the measure (6) to a one-dimensional fractional Brownian motion simulated with parameter $H_0 = 0.6$ and setting $\overline{a} = 1$ and $\overline{A} = \{2,...,20\}$. As stated by Proposition 3, when $H \neq H_0$, the diameter increases with both $|H - H_0|$ and $\overline{A}$.

III. THE ALGORITHM

Generally, once a $k$-dimensional fBm $B$ has been simulated\(^2\), the method above described can be implemented as follows

Algorithm
1) Fix a dimension of $B$.
2) For every path:

a) Calculate the increment process of lag $\overline{a}$

b) For each lag in $\overline{a} < a \leq \overline{A}$ and for each $H \in [0, 1]$ calculate:

b.1) The distance $\delta(a, H)$ between the empirical cumulative distribution of lag $a$ and $\overline{A}$

b.2) $\hat{H}_0(a) = \arg \min_{H} \delta(a, H)$

c) Estimate the self-similar parameter $\hat{H}_0$ averaging on $a$ the values $\hat{H}_0(a)$;

3) Repeat step 1) for each dimension.

Applying the algorithm to the three surrogated fBm’s of Figures 1-3, one gets the surfaces of Figures 5-7. Notice that the global minimum corresponds to the values of $H_0$ used to simulated the series.

The main problem of the above algorithm resides in the processing times required for analysing the non trivial case of $k > 1$ dimensions.

In order to speed up the estimation procedure, we improved the algorithm as follows.

Improved algorithm.
4) Fix a dimension of $B$.
5) For every path:

a) Calculate the increment process of lag $\overline{a}$

b) For lags $\overline{a}$ and $\overline{A}$ and for each $H \in [0, 1]$ calculate:

b.1) The distances $\delta(a, H)$ and $\delta(\overline{A}, H)$ between the empirical cumulative distribution of lag $a$ and, respectively, lags $\overline{a}$ and $\overline{A}$;

b.2) $\hat{H}_0(a) = \arg \min_{H} \delta(a, H)$ and $\hat{H}_0(\overline{A}) = \arg \min_{\overline{A}} \delta(a, H)$;

c) Denoted by $H_m = \min\{\hat{H}_0(a), \hat{H}_0(\overline{A})\}$ and $\hat{H}_m = \max\{\hat{H}_0(a), \hat{H}_0(\overline{A})\}$, fix $\varepsilon > 0$ and consider the interval $I = [H_m - \varepsilon, H_m + \varepsilon]$;

6) For each lag in $\overline{a} < a \leq \overline{A}$ and for each $H \in I$ calculate:

b.1) The distance $\delta(a, H)$ between the empirical cumulative distribution of lag $a$ and $\overline{A}$

b.2) $\hat{H}_0(a) = \arg \min_{H} \delta(a, H)$

7) Estimate the self-similar parameter $\hat{H}_0$ averaging on $a$ the values $\hat{H}_0(a)$;

8) Repeat step 1) for each dimension.

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\(^2\) The simulations were obtained by FracLab\® using the improved Wood and Chan algorithm [21].
The above algorithm is very simple. It is composed by a pre-estimation (step 2) and an estimation (steps 3-4). The pre-estimation basically restricts to the set $I$ the candidate self-similarity parameters $H$’s for which the test is run. As the maximal variation in terms of the diameter (but also in terms of the corresponding abscissae, due to numerical approximations) occurs in correspondence of the extremal lags, we first calculate the diameter for the minimal and the maximal lags. The points of minimum serve to define the set $I$, which is generally smaller than the whole domain $[0, 1]$. This significantly reduces the processing times, as shown in Table 1 and Figure 8, which reproduce the times in seconds required by the two algorithms for different sample sizes. Clearly, the time needed – independent on $H$ – strongly depends on the set $\mathcal{A}$. In our simulations we assumed as a rule of thumb $\mathcal{A} = \{1, 2, \ldots, 20\}$, which seems a good trade-off between the accuracy of the estimation and the processing time. The tolerance parameter $\varepsilon$ was set to 0.1.

TABLE I: TIME CONSUMPTION (IN SECONDS).

<table>
<thead>
<tr>
<th>Process length</th>
<th>Not improved</th>
<th>Improved</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>256</td>
<td>1,793</td>
<td>315</td>
<td>17.57%</td>
</tr>
<tr>
<td>512</td>
<td>4,346</td>
<td>820</td>
<td>18.87%</td>
</tr>
<tr>
<td>1024</td>
<td>12,951</td>
<td>2183</td>
<td>16.86%</td>
</tr>
<tr>
<td>2048</td>
<td>38,271</td>
<td>6879</td>
<td>17.97%</td>
</tr>
</tbody>
</table>

The calculations were performed in MatLab environment (MatLab 7.9.0.529 r 2009b) on a HP Workstation XW6200 with CPU Intel(R) XEON(TM) 3,40 GHz (two processors) and RAM 8.00 GB, with operating system environment Windows XP Professional 64 bit.

IV. CONCLUSION

An algorithm is presented that significantly improves the estimation times for a self-similar process in the general case of $k$-dimension. Further work could to be carried out about the optimal (in a numerical sense) set $\mathcal{A}$ and the tolerance parameter $\varepsilon$.

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REFERENCES

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